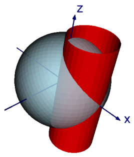
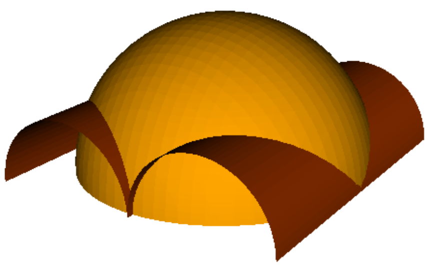


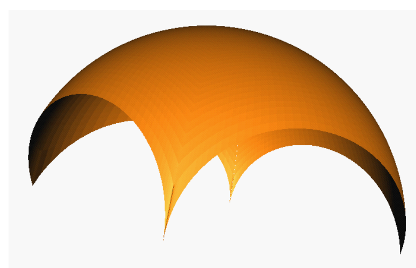
Viviani Solid with Calculus

Vinzenzo Viviani (1622-1703), Galileo's student and collaborator, was a mathematician, physicist and engineer known throughout Europe. In 1666 he became court mathematician to the Grand Duke of Tuscany, Ferdinand II de' Medici, turning down offers from Louis XIV of France and John II Casimir of Poland, among others. Viviani's work includes the first biography of Galileo Galilei, in addition, he also compiled the writings of Archimedes and Euclid.

Above all (from the point of view of this work), however, we owe him a beautiful solid with interesting properties.

In 1692, Viviani tackled the (pseudo)-architectural problem („Aenigma Geometricum de miro opificio Testudinis Quadrabilis Hemisphaericae") of constructing four equal windows on a semi-spherical vault in such a way that the area of the remaining surface is commensurate with the square of the sphere's radius. He proposed that the four windows are determined by the intersections of a hemisphere of radius R with two tangent cylinders of radius $\frac{R}{2}$, whose common part is the diameter of the sphere. Viviani called the surface left after the windows were cut out *Vela Quadrabile Fiorentina* because of its characteristic shape (*vela* means sail in Italian) and the independence of the field from π.

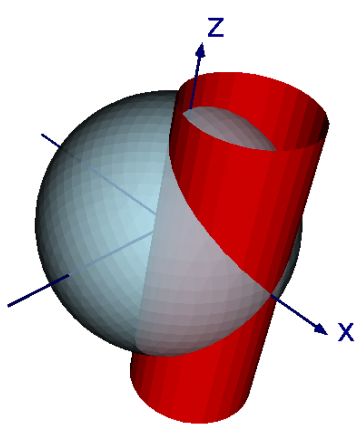




In the last picture you can see the realization of such a vault in the interior of the Basilica of San Fedele in Milan.

1. Viviani solid

A solid cut by a cylinder $x^2 + y^2 = Rx$ from a ball bounded by a sphere with equation $x^2 + y^2 + z^2 = R^2$ is called a *Viviani's solid*.

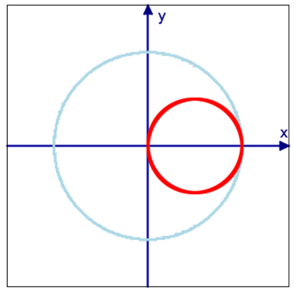


Problem 1: *Calculate the volume of Viviani's solid.*

Using the symmetry of the solid we have

$$V = 2 \cdot \iint_D \sqrt{R^2 - x^2 - y^2}\, dxdy,$$

where $z = \sqrt{R^2 - x^2 - y^2}$ is the equation of the upper hemisphere – the upper base of the Viviani solid, and the area of integration D is a disk in the Oxy plane with center at the point $\left(\frac{R}{2}, 0\right)$ and radius of length $\frac{R}{2}$. In the figure below, the disk D is bounded by a red circle with equation $x^2 + y^2 = Rx$.



The integration area and the sub-integral function indicate the usefulness of moving to polar coordinates $\begin{cases} x &= r\cos\varphi, \\ y &= r\sin\varphi. \end{cases}$

We can easily show that the polar equation of the circle $x^2 + y^2 = Rx$ is $r = R\cos\varphi$, where $\varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

However, it will be more convenient if we once again take into account the symmetry of the solid, this time with respect to the Oxz plane, and choose as the area of integration the half of the circle for $\varphi \in \left(0, \frac{\pi}{2}\right)$. Then

$$V = 4 \cdot \int_0^{\frac{\pi}{2}} d\varphi \int_0^{R\cos\varphi} \sqrt{R^2 - r^2} \cdot r\, dr.$$

The above integral can be easily calculated using the substitution method or by software like CAS:

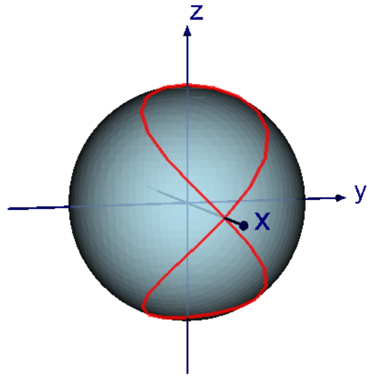
$$V = 4 \cdot \int_0^{\frac{\pi}{2}} \left(-\frac{\left(R^2 - r^2\right)^{\frac{3}{2}}}{3} \right) \Bigg|_0^{R\cos\varphi} d\varphi = 4 \cdot \int_0^{\frac{\pi}{2}} -\frac{\left(\sin^3\varphi - 1\right) R^3}{3}\, d\varphi = \frac{4}{3} R^3 \cdot \int_0^{\frac{\pi}{2}} \left(1 - \sin^3\varphi\right) d\varphi =$$

$$= \frac{4}{3} R^3 \cdot \left(-\frac{\cos^3\varphi}{3} + \cos\varphi + \varphi \right) \Bigg|_0^{\frac{\pi}{2}} = \frac{4}{3} R^3 \cdot \frac{3\,\pi - 4}{6} = \frac{2}{3}\,\pi R^3 - \frac{8}{9}\, R^3.$$

NOTICE: As we know, the volume of a hemisphere is equal to $\frac{2}{3}\,\pi R^3$. Therefore, it follows from the above calculations that the volume of its part obtained by removing the Viviani solid is equal to $\frac{8}{9}, R^3$.
This volume is therefore expressed by the monomial R^3 with a rational coefficient.

2. Viviani window

By *Viviani windows* we mean the lower and upper bases of the Viviani solid, that is, the part of the sphere $x^2 + y^2 + z^2 = R^2$ cut by the cylinder $x^2 + y^2 = Rx$. Viviani windows are bounded by a curve that is the common part of the sphere and the side surface of the cylinder. This curve is called (what a surprise ☺) the *Viviani curve*.



Problem 2: *Calculate the area of Viviani windows.*

Due to the symmetry of the solid, it is sufficient if we count the area of the upper window. It is located on the hemisphere with the equation $z = \sqrt{R^2 - x^2 - y^2}$, therefore

$$\frac{\partial z}{\partial x} = -\frac{x}{\sqrt{R^2 - y^2 - x^2}} \quad \text{i} \quad \frac{\partial z}{\partial y} = -\frac{y}{\sqrt{R^2 - y^2 - x^2}},$$

from we get

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1 + \frac{x^2}{R^2 - y^2 - x^2} + \frac{y^2}{R^2 - y^2 - x^2}} = \frac{R}{\sqrt{R^2 - y^2 - x^2}}.$$

The area of Viviani windows is therefore obtained by calculating the integral

$$S_b = 2R \cdot \iint_D \frac{dxdy}{\sqrt{R^2 - x^2 - y^2}},$$

where again the integration region D is a disk bounded by a circle with the equation $x^2 + y^2 = Rx$.

Moving to polar coordinates we get

$$S_b = 2R \cdot \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{R\cos\varphi} \frac{rdr}{\sqrt{R^2 - r^2}} = 4R \cdot \int_0^{\frac{\pi}{2}} d\varphi \int_0^{R\cos\varphi} \frac{rdr}{\sqrt{R^2 - r^2}},$$

whenceforth

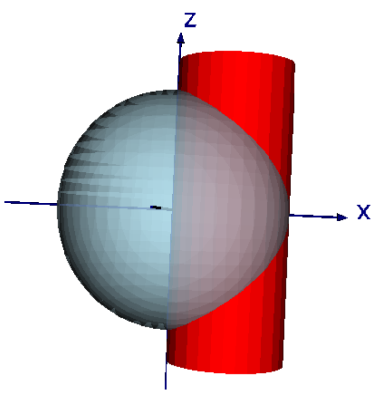
$$S_b = 4R \cdot \int_0^{\frac{\pi}{2}} \left(-\sqrt{R^2 - r^2}\right)\Big|_0^{R\cos\varphi} d\varphi = 4R \cdot \int_0^{\frac{\pi}{2}} \left(R - R\sqrt{1 - \cos^2\varphi}\right) d\varphi = 4R^2 \cdot \int_0^{\frac{\pi}{2}} (1 - \sin\varphi)\, d\varphi =$$

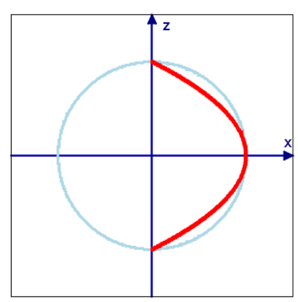
$$= 4R^2 \cdot \left(\varphi + \cos\varphi\right)\Big|_0^{\frac{\pi}{2}} = 4R^2 \cdot \left(\frac{\pi}{2} - 1\right) = 2\pi R^2 - 4R^2.$$

NOTICE: The area of the hemisphere is equal to $2\pi R^2$, so the area of that part of the hemisphere that remains after removing the Viviani windows is equal to $4R^2$. Thus, it is expressed by a monomial R^2 with a rational (and in this case even integer) coefficient – a phenomenon we have already observed by calculating the volume of the Viviani solid.

3. Lateral surface of the Viviani solid

Problem 3: *Calculate the area of the side surface of the Viviani solid, that is, the area of the cylinder $x^2 + y^2 = Rx$ contained inside the sphere $x^2 + y^2 + z^2 = R^2$.*





We will use the formula used in the solution of problem 2. To do this, let us note that the side surface of the Viviani solid is symmetric with respect to the Oxz plane, and its front part is the graph of the function $y = \sqrt{Rx - x^2}$ of the variables x and z, therefore

$$\frac{\partial y}{\partial x} = \frac{R - 2x}{2\sqrt{xR - x^2}} \quad \text{i} \quad \frac{\partial y}{\partial z} = 0,$$

and further

$$\sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2} = \sqrt{1 + \frac{(R - 2x)^2}{4(xR - x^2)} + 0} = \frac{R}{2\sqrt{x(R - x)}}.$$

The area of the side surface of the Viviani solid is therefore obtained by calculating the integral

$$S_l \;=\; 2\cdot \iint_D \frac{R}{2\sqrt{x\,(R-x)}}\,dxdz,$$

where this time the area of integration D lies in the Oxz plane and is bounded by the Oz axis and the (red) parabola with the equation $Rx + z^2 = R^2$ (see figure above).

This time the transition to polar coordinates is not necessary, so (we use symmetry!)

$$S_l \;=\; 2R\cdot\int_0^R dx\int_0^{\sqrt{R^2-Rx}} \frac{1}{\sqrt{x\,(R-x)}}\,dz \;=\; 2R\cdot\int_0^R \left.\frac{z}{\sqrt{x\,(R-x)}}\right|_0^{\sqrt{R^2-Rx}} dx \;=\; 2R\cdot\int_0^R \frac{\sqrt{R^2-x\,R}}{\sqrt{x\,(R-x)}}\,dx \;=$$

$$=\; 2R\sqrt{R}\cdot\int_0^R \frac{1}{\sqrt{x}}\,dx \;=\; 4R\sqrt{R}\cdot\left.\sqrt{x}\right|_0^R \;=\; 4R^2.$$

NOTICE: Thus, this area again does not contain the number π!